Epsilon – Delta Proof by Andy Miller
Frederick Community College

Project objectives:
- Understand the concept of epsilon-delta proof of a limit for functions of one variable.
- Understand the concept of uniform continuity of functions of one variable.
- Apply the epsion-delta proof to functions of two variables (multi-variable calculus).
- Use the epsilon-delta proof of a limit to prove continuity of various functions.

Epsilon Delta Definition:
For each $\epsilon > 0$, there exists $\delta > 0$ such that $|x - x_0| < \delta$ implies that $|f(x) - f(x_0)| < \epsilon$.

Continuity of a linear function
For $f(x) = 4x + 3$, prove that $\lim_{x \to 2}(4x + 3) = 11$.

Proof: Let $\epsilon > 0$ be given, then choose $\delta = \frac{\epsilon}{4}$. Then if $|x - 2| < \delta$, we have

$$|(4x + 3) - 11| = |4x - 8| = 4|x - 2| < 4 \cdot \frac{\epsilon}{4} = \epsilon$$

Continuity of a quadratic function
For $f(x) = x^2$, prove that $\lim_{x \to 3}(x^2) = 9$.

Proof: Let $\epsilon > 0$ be given, then choose $\delta = \min \{1, \frac{\epsilon}{7} \}$. Then if $|x - 3| < \delta$, we have

$$|x^2 - 9| = |(x - 3)(x + 3)| < 7|x - 3| < 7 \cdot \frac{\epsilon}{7} = \epsilon$$

Uniform continuity of a cubic function
For $f(x) = x^3$, prove that $f$ is uniformly continuous on a closed interval $[-a, a]$.

Proof: Let $\epsilon > 0$ be given, then choose $\delta = \min \{1, \frac{\epsilon}{3a^2} \}$. Then if $|x_1 - x_2| < \delta$, we have

$$|x_1^3 - x_2^3| = |(x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2)| \leq (|x_1 - x_2|)(|x_1^2| + a^2 + |x_2^2|) \leq |x_1 - x_2|(|a^2| + a + |a^2|) < \delta \cdot (3a^2) < \epsilon$$

Limit of a function of two variables
For $f(x, y) = \frac{4xy^4}{x^2 + y^2}$, prove that $\lim_{(x,y) \to (0,0)} f(x, y) = 0$.

Proof: Let $\epsilon > 0$ be given, then choose $\delta = \frac{\epsilon}{4}$. Then if $\sqrt{x^2 + y^2} < \delta$, we have

$$\left|\frac{4xy^4}{x^2 + y^2} - 0\right| = \frac{4\sqrt{x^2} \cdot y^2}{x^2 + y^2} \leq \frac{4\sqrt{x^2 + y^2} \cdot y^2}{x^2 + y^2} = \frac{4y^2}{\sqrt{x^2 + y^2}} \leq \frac{4(x^2 + y^2)}{\sqrt{x^2 + y^2}} \leq 4 \cdot \delta = 4 \cdot \frac{\epsilon}{4} = \epsilon$$

The problem that started it all
For $f(x, y) = \frac{x^2y^3}{2x^2 + y^2}$, prove that $\lim_{(x,y) \to (0,0)} f(x, y) = 0$.

Proof: Let $\epsilon > 0$ be given, then choose $\delta = \sqrt{2}\epsilon$. Then if $\sqrt{x^2 + y^2} < \delta$, we have

$$\left|\frac{x^2y^3}{2x^2 + y^2} - 0\right| \leq \frac{(x^2 + y^2)y^3}{2x^2 + y^2} \leq \frac{(x^2 + y^2)(x^2 + y^2)y}{2x^2 + y^2} \leq \frac{(x^2 + y^2)(x^2 + y^2)\sqrt{x^2 + y^2}}{2x^2 + y^2} \leq \frac{\delta^3 \cdot \sqrt{2}\epsilon}{2 \delta^2} = \frac{\delta^3}{2} = \frac{(\sqrt{2}\epsilon)^3}{2} = \frac{2\epsilon}{2} = \epsilon$$